11. 3. 62.

Dear Frank,

Sorry to have been so long in returning your paper. I had hoped to do it last weekend, but I have had a beast of a cold which has been hovering on the brink of bronchitis and/or tonsillitis, and I haven't felt like highbrow mathematics. Today I have spent the day in bed in an effort to shake it off, and have taken the opportunity to go through your paper. General impression is very nice indeed. I have indicated in the margin one or two minor obscurities or misprints, and have added one general comment on your use of "thus" (see top of p. 1 of your MS.) I was rather amused to see that you follow the classical analysts in writing both f and $f(x)$ for the function whose value at x is $f(x)$. More seriously, I felt that there were one or two places where the notation $x \to f(x)$ for the function would have clarified your statements.

I enclose an offprint of the Gazette paper; I'm sad to say I don't use any of it for undergraduate teaching – my lectures on the Riemann integral are trimmed to the minimum.

You asked about going from $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$ for $a, b, a+b$ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ to the same relation for general a, b. The problem certainly exists, but I don't know an elegant method of doing it. My answer, I think, is that I don't seriously recommend the integral def^n of the trig fns – the arbitrary extension of the domain of the fns by periodicity is highly artificial.

I think the most natural treatment, which one can take reasonably early in the course, is to define sin as that sol^n of $y'' + y = 0$ which is twice-diff^ble for all x and satisfies ~~sin~~ $\sin 0 = 0$, $\sin' 0 = 1$. Then define $\cos = \sin'$. Assume right from the beginning that the eq^n $y''+y=0$ has at least one non-trivial solution (this can be justified later by uniform cgce of power series) and carry on from there. Basic steps are

(1) Uniqueness. $\iff$ If f is twice-diff'ble and $f''(x)+f(x)=0$ for all x, and $f(0)=f'(0)=0$, then $f\equiv 0$. Proof: Let $g=f^2+f'^2$. Then $g'\equiv 0$, so $f\equiv 0$.

(2) sin is ~~odd~~, cos is odd. Apply (1) to $f(x) = \sin' x + \sin'(-x)$.

(3) Add^n theorems. Apply (1) to $f(x) = \sin(x+y) - \sin x \cos y - \cos x \sin' y$.

[This is essentially Ron Clarke's method you mentioned to me once ~~Goes back to~~ Also in Verblunsky's book on f^ns of a real variable].

(4) $\cos^2 x + \sin^2 x = 1$, $|\cos x| \leq 1$, $|\sin x| \leq 1$, $\forall x$. By (3) & (2).

(5) Periodicity. All boils down to proving $\exists$ a zero of cos. Nicest proof goes via the

LEMMA. If $f(x)\to l$ as $x\to+\infty$, & $f''(x)=O(1)$ as $x\to+\infty$, then $f'(x)\to 0$ as $x\to+\infty$.

(For a proof, see Littlewood's Mathematician's Miscellany, on use of pictures in proofs)

If $\cos x \neq 0$, then $\cos x > 0$ $\forall x$, so sin is (strictly) $\uparrow$, whence $\sin x \to l$ as $x\to+\infty$. Hence $\cos x \to 0$ as $x\to+\infty$, by the Lemma, so $l=1$, since $\cos^2+\sin^2=1$. Now apply the Lemma to cos; we get $\sin x\to 0$ as $x\to+\infty$, contradicting $l=1$.

Since Christmas I have been working quite seriously on a textbook of elementary analysis, and I produced the above treatment a week or so ago whilst dealing with a chapter on derivatives. I had considered and rejected a number of treatments of the trig f^ns, and this seems the best yet. The thought of a book arose from a tentative enquiry about one from McGraw Hill (who later lost interest when they discovered the potential market for a book for Honours math^{cs} only), and I sat down to see how long it was likely to take. ~~My present aim is to try to get it done by the end of the summer~~. I have gone all modern, and firmly reattached a function to its domain, and tried to avoid 'the function $f(x)$'. ~~But~~ It is a hell of a job to do so without making it look merely pedantic, but the necessity to do so becomes clearer when one considers the derivative of the function $x \to \log\log \sin x$.

My present aim is to include diffn of fns of 1 and 2 variables (but excluding the defn of a differential – enquiries round our department produced ~~six different defns~~ at least three being inequivalent), Riemann integn for fns of 1 variable, convergence and uniform convergence, and general theory of metric spaces (open, closed, compact, complete, connected). I want to avoid multiple integration and vector calculus, for it would make the book too long to include them. ~~So far I have about 140 pages roughly done, and at least as much again still to do.~~

I'll leave all my personal news till I see you. We are hoping to go away for four days, March 29th – April 1st inclusive, up to Corbridge, on the upper Tyne. We also have our reading party at Burton Manor on April 15 – 19th, and then Jean & Joan's Dad here for the Easter week-end. If you can time your visit to Liverpool to miss all these, we shall be delighted to see you and to put you up. Otherwise I'll see you at the Colloquium. If you are using both Glen Eyre & Chamberlayne for the Colloquium, I would prefer to be beside you in Glen Eyre, but if you are using Chamberlayne only, then ~~for the sake of appearances~~ (since I hope to get some of my expenses paid to come) I think I had better say Chamberlayne.